

Chapter review 2

1 a Parametric equations: $\cos \theta = \frac{x}{4}$ and $\sin \theta = \frac{y}{9}$

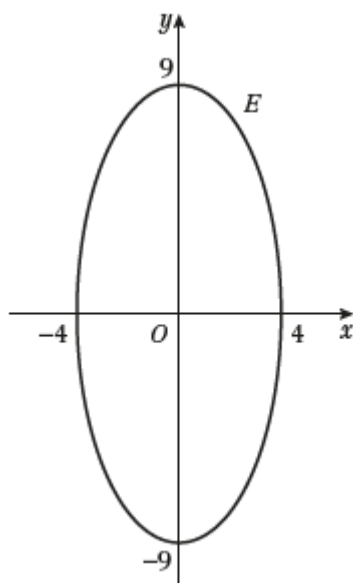
$$\cos^2 \theta + \sin^2 \theta \equiv 1$$

Substituting the values for $\cos \theta$ and $\sin \theta$ in the equation for ellipse E

gives the Cartesian equation: $\frac{x^2}{4^2} + \frac{y^2}{9^2} = 1$

b Comparing with the equation in its standard form, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = 4$ and $b = 9$

So a sketch of E is:



c Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{9 \cos \theta}{-4 \sin \theta}$

The gradient of the normal is $\frac{4 \sin \theta}{9 \cos \theta}$

Use $y - y_0 = m(x - x_0)$ to get the equation: $y - 9 \sin \theta = \frac{4 \sin \theta}{9 \cos \theta} (x - 4 \cos \theta)$

$$\Rightarrow 9y \cos \theta - 81 \sin \theta \cos \theta = 4 \sin \theta (x - 4 \cos \theta)$$

So the equation of the normal is $4x \sin \theta - 9y \cos \theta = -65 \sin \theta \cos \theta$

2 a Parametric equations: $x = \pm 2 \cosh t$ and $y = 5 \sinh t$

$$\cosh^2 t - \sinh^2 t \equiv 1$$

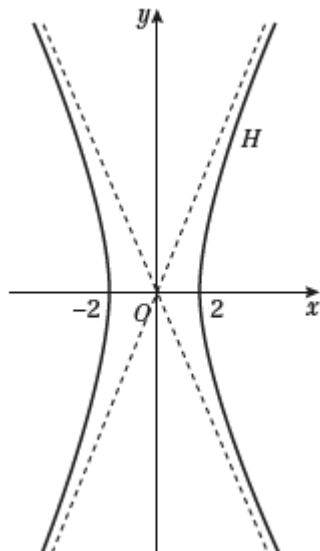
Substituting the values for $\cosh t$ and $\sinh t$ in the equation for the hyperbola H

gives the Cartesian equation: $\frac{x^2}{2^2} - \frac{y^2}{5^2} = 1$

- 2 b Comparing with the equation in its standard form, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a = 2$ and $b = 5$

So the asymptotes are the lines $y = \pm \frac{5}{2}x$ and the hyperbola cuts the x -axis at ± 2

A sketch is the following:



- c Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5 \cosh t}{2 \sinh t}$

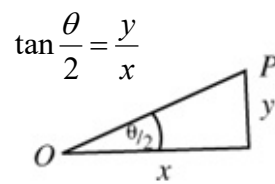
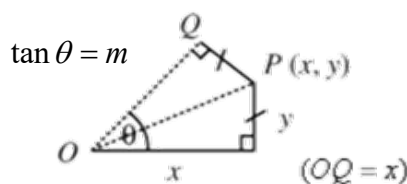
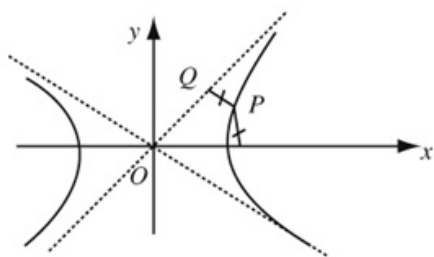
Use $y - y_0 = m(x - x_0)$ to get the equation: $y - 5 \sinh t = \frac{5 \cosh t}{2 \sinh t}(x - 2 \cosh t)$

$$\Rightarrow 2y \sinh t + 10 = 5x \cosh t$$

- 3 a Asymptotes are $y = \pm \frac{b}{a}x$ so $m = \frac{b}{a} \Rightarrow b = am$

Substituting in the equation for the hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{a^2 m^2} = 1$

3 b Let Q be the point on the asymptote.



$$\text{Using } \tan \theta = \frac{2 \tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)} \Rightarrow m = \frac{2\left(\frac{y}{x}\right)}{1 - \left(\frac{y^2}{x^2}\right)} = \frac{2xy}{(x^2 - y^2)} \quad (1)$$

But P lies on the hyperbola so from part a, $x^2 m^2 - y^2 = a^2 m^2$

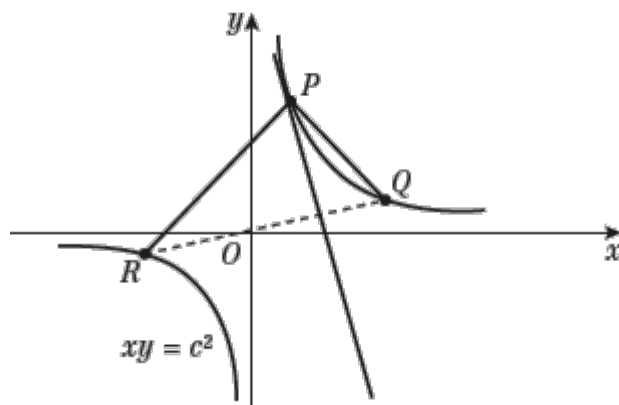
$$\text{So } m^2 = \frac{y^2}{x^2 - a^2} \quad (2)$$

$$\text{Using (1)}^2 \text{ and (2)} \quad \frac{4x^2 y^2}{(x^2 - y^2)^2} = \frac{y^2}{x^2 - a^2}$$

$$\text{So } 4x^2(x^2 - a^2) = (x^2 - y^2)^2$$

$$4 \text{ a Gradient of chord } PQ = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{\cancel{c}(q-p)}{pq \cancel{c}(p-q)} = \frac{-1}{pq}$$

b



$$P\left(cp, \frac{c}{p}\right); Q\left(cq, \frac{c}{q}\right); R\left(cr, \frac{c}{r}\right)$$

$$\text{Gradient of } PQ = -\frac{1}{pq}$$

$$\text{Gradient of } PR = -\frac{1}{pr}$$

$$\text{If } \angle PQR = 90^\circ \Rightarrow -\frac{1}{pq} \times -\frac{1}{pr} = -1 \Rightarrow -1 = p^2 qr \quad (1)$$

To find gradient of tangent at P , let $q \rightarrow p$ for chord PQ

$$\text{Gradient of tangent at } P \text{ is } -\frac{1}{p^2}$$

$$\text{Gradient of chord } RQ \text{ is } -\frac{1}{qr}$$

$$\frac{-1}{qr} \times -\frac{1}{p^2} = \frac{1}{p^2 qr}$$

$$\text{But from (1) } p^2 qr = -1$$

So gradient of tangent at $P \times$ gradient of $QR = -1$

Therefore the tangent at P is perpendicular to chord QR .

$$5 \text{ a } y = ct^{-1}, x = ct \Rightarrow \frac{dy}{dx} = \frac{-ct^{-2}}{c} = -\frac{1}{t^2}$$

$$\text{Equation of tangent is: } y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$\Rightarrow yt^2 - ct = -x + ct \quad \text{or} \quad t^2y + x = 2ct$$

$$b \text{ Let } S\left(cs, \frac{c}{s}\right) \text{ and } T\left(ct, \frac{c}{t}\right) \text{ be two points on the hyperbola } xy = 16 \text{ (} c = 4 \text{)}$$

$$\text{So tangent at } S \text{ is } s^2y + x = 2cs$$

Using the equations for the tangent at T and S , intersection of tangents is:

$$(t^2 - s^2)y = 2c(t - s)$$

$$y = \frac{2c}{t + s}$$

$$\text{So } x = 2ct - \frac{2ct^2}{t + s} = \frac{2cts}{t + s}$$

$$\text{When } c = 4 \text{ the point of intersection is } \left(\frac{8ts}{t + s}, \frac{8}{t + s}\right)$$

The tangents intersect at $(-3, 3)$:

$$x = -3: -3 = \frac{8ts}{t + s} \Rightarrow -3(t + s) = 8ts$$

$$y = 3: 3 = \frac{8}{t + s} \Rightarrow 3(t + s) = 8$$

$$\text{So } ts = -1 \Rightarrow t = -\frac{1}{s}$$

$$\text{Substituting for } t: 3\left(s - \frac{1}{s}\right) = 8 \Rightarrow 3s^2 - 8s - 3 = 0$$

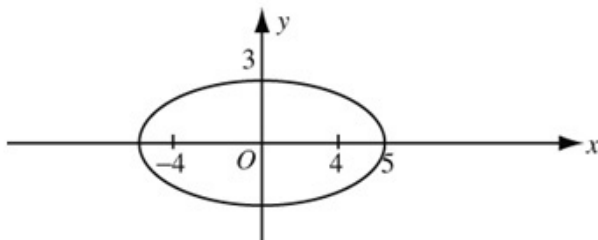
$$\text{Factorising: } (3s + 1)(s - 3) = 0$$

$$\text{So } s = -\frac{1}{3} \text{ or } s = 3; \quad t = 3 \text{ or } t = -\frac{1}{3}$$

$$\text{The points } S \text{ and } T \text{ are } \left(-\frac{4}{3}, -12\right) \text{ and } \left(12, \frac{4}{3}\right)$$

$$6 \text{ a } 9x^2 + 25y^2 = 225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$\therefore a = 5, b = 3$$



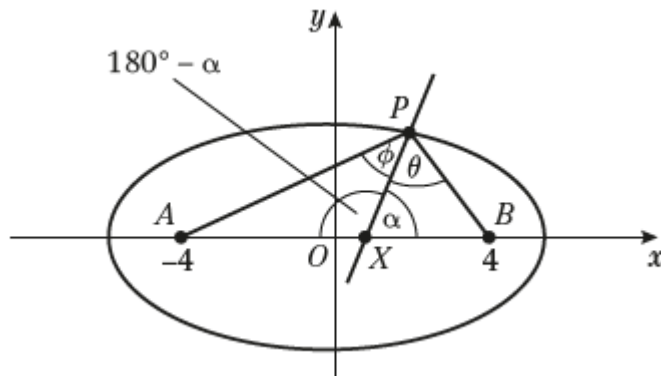
$$b^2 = a^2(1 - e^2) \Rightarrow 9 = 25(1 - e^2) \Rightarrow e^2 = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

The foci are at $\left(\pm \frac{a}{e}, 0\right)$, which is $(\pm 4, 0)$, so A and B are the foci.

From the properties of an ellipse, $PS + PS' = 2a = 10$

$$\text{So } PA + PB = 10$$

6 b



Normal at P is: $5x \sin t - 3y \cos t = 16 \cos t \sin t$

X is when $y = 0$, i.e. $x = \frac{16}{5} \cos t$

$$PB^2 = (5 \cos t - 4)^2 + (3 \sin t)^2 = 25 \cos^2 t - 40 \cos t + 16 + 9 \sin^2 t \\ = 16 \cos^2 t - 40 \cos t + 25 = (4 \cos t - 5)^2$$

$$\Rightarrow PB = 5 - 4 \cos t \quad \text{and} \quad PA = 10 - PB = 5 + 4 \cos t$$

$$AX = 4 + \frac{16}{5} \cos t, \quad BX = 4 - \frac{16}{5} \cos t$$

If PX bisects angle APB , then angle $\theta =$ angle ϕ

Consider sine rule on $\triangle PAX$: $\sin \phi = \frac{\sin(180^\circ - \alpha)AX}{AP} = \frac{\sin \alpha(4 + \frac{16}{5} \cos t)}{5 + 4 \cos t} = \frac{4}{5} \sin \alpha$

Consider sine rule on $\triangle PBX$: $\sin \theta = \frac{\sin \alpha BX}{PB} = \frac{\sin \alpha(4 - \frac{16}{5} \cos t)}{5 - 4 \cos t} = \frac{4}{5} \sin \alpha$

So $\sin \theta = \sin \phi$ and since both angles are acute, the normal bisects APB .

7 a $y = ct^{-1}$, $x = ct \Rightarrow \frac{dy}{dx} = \frac{-ct^{-2}}{c} = -\frac{1}{t^2}$

Equation of tangent is: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$$\Rightarrow yt^2 - ct = -x + ct \quad \text{or} \quad t^2 y + x = 2ct$$

b Gradient of tangent is $-\frac{1}{t^2}$, so gradient of OP is t^2

Equation of OP is $y = t^2 x$

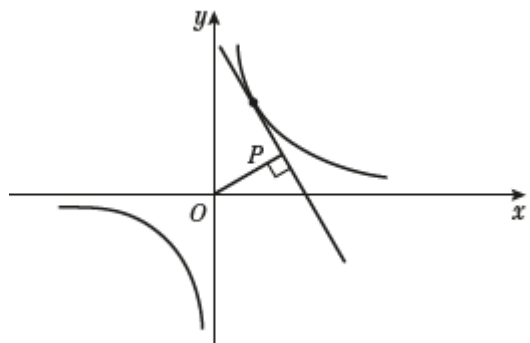
Equation of tangent is $t^2 y = 2ct - x$

Solving $t^4 x = 2ct - x$

$$\Rightarrow x = \frac{2ct}{1+t^4}, \quad y = \frac{2ct^3}{1+t^4}$$

$$x^2 + y^2 = \frac{4c^2 t^2 + 4c^2 t^6}{(1+t^4)^2} = \frac{4c^2 t^2 (1+t^4)}{(1+t^4)^2}$$

$$\Rightarrow \left. \begin{aligned} (x^2 + y^2)^2 &= \frac{16c^4 t^4}{(1+t^4)^2} \\ xy &= \frac{4c^2 t^4}{(1+t^4)^2} \end{aligned} \right\} \Rightarrow (x^2 + y^2)^2 = 4c^2 xy$$



8 a Gradient $OP = \frac{2ap}{ap^2} = \frac{2}{p}$, gradient of $OQ = \frac{2}{q}$

Since OP and OQ are perpendicular,

$$\frac{4}{pq} = -1 \Rightarrow pq = -4$$

b Normal at Q is $y + xq = aq^3 + 2aq$

c Normal at P is $y + xp = ap^3 + 2ap$

Solving the equations for the tangents simultaneously:

$$x(q - p) = a(q^3 - p^3) + 2a(q - p)$$

$$x(\cancel{q - p}) = a(\cancel{q - p})(q^2 + qp + p^2) + 2a(\cancel{q - p})$$

$$x = a(q^2 + p^2 + qp + 2)$$

$$y = ap^3 + \cancel{2ap} - apq^2 - ap^3 - aqp^2 - \cancel{2ap} \Rightarrow y = -apq(q + p)$$

But if $pq = -4$ then R is $(ap^2 + aq^2 - 2a, 4a(p + q))$

d Express $p^2 + q^2$ as $(p + q)^2 - 2pq$

Then $X = a((p + q)^2 - 2pq - 2) = a((p + q)^2 + 6)$

$$Y = 4a(p + q) \Rightarrow p + q = \frac{Y}{4a}$$

$$\Rightarrow X = a\left(\frac{Y^2}{16a^2} + 6\right)$$

$$X - 6a = \frac{Y^2}{16a} \Rightarrow Y^2 = 16aX - 96a^2$$

9 $y = mx + c$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$

$$\Rightarrow x^2(b^2 + a^2m^2) + 2a^2mxc + a^2(c^2 - b^2) = 0$$

For a tangent the discriminant = 0:

$$4a^4m^2c^2 = 4(b^2 + a^2m^2)a^2(c^2 - b^2)$$

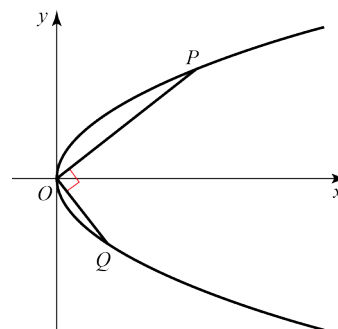
$$a^2m^2c^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$$

$$a^2m^2b^2 + b^4 = b^2c^2$$

$$\Rightarrow c^2 = a^2m^2 + b^2$$

$$c = \pm\sqrt{a^2m^2 + b^2}$$

So the lines $y = mx \pm \sqrt{a^2m^2 + b^2}$ are tangents for all m .



10 Chord PQ has gradient $\frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{c(q-p)}{pqc(p-q)} = -\frac{1}{pq}$

If gradient = 1, then $pq = -1$

Tangent at P is $p^2y + x = 2cp$

Tangent at Q is $q^2y + x = 2cq$

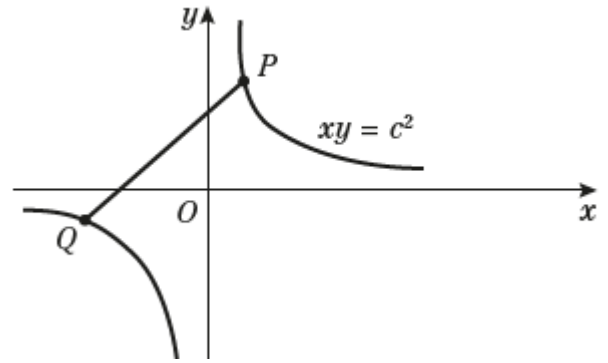
Intersection: $(p^2 - q^2)y = 2c(p - q) \Rightarrow y = \frac{2c}{p+q}$

$$\Rightarrow x = 2cp - \frac{2cp^2}{p+q} = \frac{2cpq}{p+q}$$

So the point of intersection R is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

But $pq = -1$, so R is $x = \frac{-2c}{p+q}$, $y = \frac{2c}{p+q}$, i.e. $y = -x$

The locus of R is the line $y = -x$



11 a Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4 \cos \theta}{-6 \sin \theta} = -\frac{2 \cos \theta}{3 \sin \theta}$

Line l_1 has equation: $y - 4 \sin \theta = -\frac{2 \cos \theta}{3 \sin \theta}(x - 6 \cos \theta)$

$$3y \sin \theta - 12 \sin^2 \theta = -2x \cos \theta + 12 \cos^2 \theta$$

$$2x \cos \theta + 3y \sin \theta = 12 \cos^2 \theta + 12 \sin^2 \theta$$

$$2x \cos \theta + 3y \sin \theta = 12$$

b $y = 0$: $2x \cos \theta = 12 \Rightarrow x = \frac{6}{\cos \theta}$

$x = 0$: $3y \sin \theta = 12 \Rightarrow y = \frac{4}{\sin \theta}$

So A has coordinates $\left(\frac{6}{\cos \theta}, 0\right)$ and B has coordinates $\left(0, \frac{4}{\sin \theta}\right)$

Midpoint of AB has coordinates $\left(\frac{3}{\cos \theta}, \frac{2}{\sin \theta}\right)$, so $x = \frac{3}{\cos \theta}$ and $y = \frac{2}{\sin \theta}$

Using $\cos^2 \theta + \sin^2 \theta = 1$: $\left(\frac{3}{x}\right)^2 + \left(\frac{2}{y}\right)^2 = 1$

The locus of the midpoint of AB is $\frac{9}{x^2} + \frac{4}{y^2} = 1$

12 a Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5 \cos \theta}{-13 \sin \theta}$

An equation for the tangent line l_1 is: $y - 5 \sin \theta = -\frac{5 \cos \theta}{13 \sin \theta}(x - 13 \cos \theta)$

$$13y \sin \theta - 65 \sin^2 \theta = -5x \cos \theta + 65 \cos^2 \theta$$

$$13y \sin \theta + 5x \cos^2 \theta = 65$$

b The point A has x -coordinate equal to 0, so its coordinates are $\left(0, \frac{5}{\sin \theta}\right)$

Line l_2 is perpendicular to l_1 so has gradient $\frac{13 \sin \theta}{5 \cos \theta}$

The equation of the line l_2 is given by: $y - \frac{5}{\sin \theta} = \frac{13 \sin \theta}{5 \cos \theta}x$

$$5y \sin \theta \cos \theta - 25 \cos \theta = 13x \sin^2 \theta$$

c Using $a = 13$ and $b = 5$, $b^2 = a^2(1 - e^2) \Rightarrow 25 = 169(1 - e^2) \Rightarrow e = \sqrt{1 - \frac{25}{169}}$

So the eccentricity of the ellipse is $e = \frac{12}{13}$

l_2 cuts the x -axis at $(-ae, 0)$, which is $(-12, 0)$.

Substitute this into the equation of l_2 :

$$-25 \cos \theta = -156 \sin^2 \theta$$

$$25 \cos \theta = 156(1 - \cos^2 \theta)$$

$$156 \cos^2 \theta + 25 \cos \theta - 156 = 0$$

The solutions of this equation are $\cos \theta = \frac{-25 \pm 313}{312}$

This gives either $\cos \theta = -\frac{338}{312}$, which can't be a cosine, or $\cos \theta = \frac{12}{13} = e$

13 a Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{8 \sec^2 \theta}{4 \sec \theta \tan \theta} = \frac{2}{\sin \theta}$

The gradient of the normal l_1 is $-\frac{\sin \theta}{2}$

Use $y - y_0 = m(x - x_0)$ to get the equation: $y - 8 \tan \theta = -\frac{\sin \theta}{2}(x - 4 \sec \theta)$

$$2y - 16 \tan \theta = -x \sin \theta + 4 \tan \theta$$

$$x \sin \theta + 2y = 20 \tan \theta$$

$$13 \text{ b } y = 0: x \sin \theta = 20 \tan \theta \Rightarrow x = 20 \sec \theta$$

$$x = 0: 2y = 20 \tan \theta \Rightarrow y = 10 \tan \theta$$

So A has coordinates $(20 \sec \theta, 0)$ and B has coordinates $(0, 10 \tan \theta)$

Midpoint of AB has coordinates $(10 \sec \theta, 5 \tan \theta)$, so $x = 10 \sec \theta$ and $y = 5 \tan \theta$

$$\text{Using } \sec^2 \theta - \tan^2 \theta \equiv 1: \quad \frac{x^2}{10^2} - \frac{y^2}{5^2} = 1$$

The locus of the midpoint is $\frac{x^2}{100} - \frac{y^2}{25} = 1$, which is the equation of a hyperbola.

$$14 \text{ a } \text{ Use the chain rule to find the gradient of the tangent: } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b \cos t}{-a \sin t}$$

The gradient of the normal is $\frac{a \sin t}{b \cos t}$

$$\text{An equation for the line is: } y - b \sin t = \frac{a \sin t}{b \cos t} (x - a \cos t)$$

$$by \cos t - b^2 \sin t \cos t = ax \sin t - a^2 \sin t \cos t$$

$$ax \sin t - by \cos t = (a^2 - b^2) \sin t \cos t$$

$$\text{b } y = 0: ax \sin t = (a^2 - b^2) \sin t \cos t \Rightarrow x = \left(\frac{a^2 - b^2}{a} \right) \cos t$$

$$x = 0: by \cos t = -(a^2 - b^2) \sin t \cos t \Rightarrow y = - \left(\frac{a^2 - b^2}{b} \right) \sin t$$

So M has coordinates $\left(\frac{a^2 - b^2}{a} \cos t, 0 \right)$ and N has coordinates $\left(0, -\frac{a^2 - b^2}{b} \sin t \right)$

Midpoint of MN has coordinates: $\left(\frac{a^2 - b^2}{2a} \cos t, -\frac{a^2 - b^2}{2b} \sin t \right)$

$$x = \frac{a^2 - b^2}{2a} \cos t \Rightarrow \cos t = \frac{2ax}{a^2 - b^2}$$

$$y = -\frac{a^2 - b^2}{2b} \sin t \Rightarrow \sin t = -\frac{2by}{a^2 - b^2}$$

$$\text{Using } \cos^2 t + \sin^2 t \equiv 1: \quad \frac{4a^2 x^2}{(a^2 - b^2)^2} + \frac{4b^2 y^2}{(a^2 - b^2)^2} = 1$$

$$\Rightarrow 4a^2 x^2 + 4b^2 y^2 = (a^2 - b^2)^2$$

The locus described by the midpoint of MN is $4a^2 x^2 + 4b^2 y^2 = (a^2 - b^2)^2$

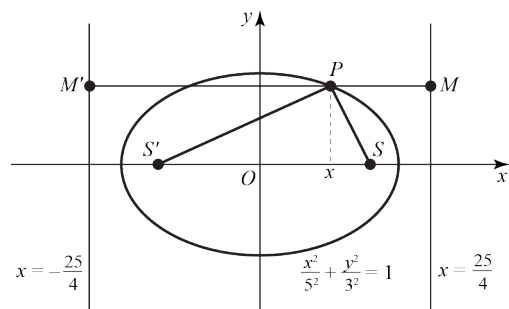
$$15 \ a = 5 \text{ and } b = 3$$

$$\text{Using } b^2 = (1 - e^2)a^2 : 9 = 25(1 - e^2) \Rightarrow e^2 = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

The directrices of the ellipse are the lines of equation $x = \pm \frac{25}{4}$

P is the point (x, y) .

By definition, for any point P on an ellipse, the ratio of the distance of P from the focus of the ellipse to the distance of P from the directrix is constant, called the eccentricity e .



Let M be the point on the directrix $x = \frac{25}{4}$ where $PS = ePM$

Let M' be the point on the directrix $x = -\frac{25}{4}$ where $PS' = ePM'$

PM and PM' are parallel to the x -axis.

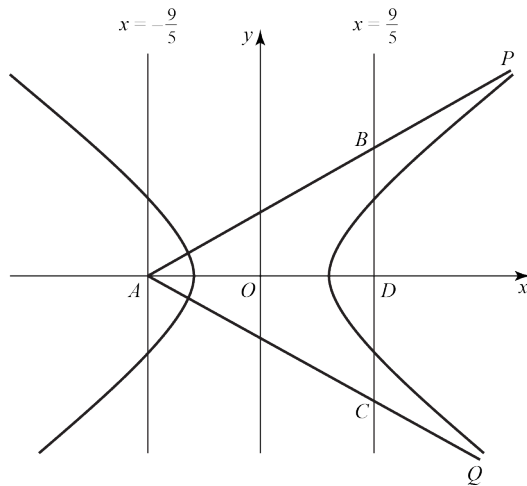
$$PM' = \frac{25}{4} + x \text{ and } PM = \frac{25}{4} - x$$

$$\text{So } PS + PS' = ePM + ePM' = \frac{4}{5} \times \frac{50}{2} = 10$$

$$16 a = 3 \text{ and } b = 4, \text{ so } b^2 = a^2(e^2 - 1) \Rightarrow e^2 = \frac{25}{9} \Rightarrow e = \frac{5}{3}$$

Directrices are at $\pm \frac{a}{e}$ so they are $x = \pm \frac{9}{5}$

Let P and Q be on the right-hand side of the hyperbola. The tangents at P and Q meet the directrix $x = -\frac{9}{5}$ at A ($y = 0$) and the directrix $x = \frac{9}{5}$ at B and C . The point D is where the directrix $x = \frac{9}{5}$ crosses the x -axis.



The base of triangle ABC is the distance BC , while the height is the distance between the directrices. The parametric equation for H is $x = 3 \sec t$, $y = 4 \tan t$

Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 \sec t}{3 \tan t}$

So the tangent to H has the equation: $y - 4 \tan t = \frac{4 \sec t}{3 \tan t}(x - 3 \sec t)$

$$3y \tan t - 12 \tan^2 t = 4x \sec t - 12 \sec^2 t$$

$$3y \tan t + 12(\sec^2 t - \tan^2 t) = 4x \sec t$$

$$3y \tan t + 12 = 4x \sec t$$

$$4x \sec t - 3y \tan t = 12$$

Tangents meet at A , which is $\left(-\frac{9}{5}, 0\right)$, so let $x = -\frac{9}{5}$, $y = 0$

$$\Rightarrow \frac{36}{5} \sec t = 12 \quad \text{so} \quad \cos t = \frac{3}{5} \Rightarrow t = \pm 0.927 \dots$$

$$\sec \pm 0.927 \dots = \frac{5}{3}, \quad \tan \pm 0.927 \dots = \pm \frac{4}{3}$$

The gradient of the tangent with positive slope is $\frac{4 \sec t}{3 \tan t} = \frac{4 \times \frac{5}{3}}{3 \times \frac{4}{3}} = \frac{5}{3}$

B is on the tangent, so $\frac{BD}{AD} = \frac{5}{3} \Rightarrow BD = \frac{5}{3} \times \frac{18}{5} = 6$

By symmetry, BD is half of BC , so the area of ABC is $BD \times AD = 6 \times \frac{18}{5} = \frac{108}{5}$

Challenge

Let $P = (a \cos \theta, b \sin \theta)$

The focus has coordinates $(ae, 0)$, so the distance PS^2 is:

$$PS^2 = (ae - a \cos \theta)^2 + b^2 \sin^2 \theta$$

$$= a^2 e^2 - 2a^2 e \cos \theta + a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

The equation of the normal is $ax \sin \theta - by \cos \theta = (a^2 - b^2) \sin \theta \cos \theta$

This intersects the x -axis at $x = \frac{(a^2 - b^2) \cos \theta}{a} = ae^2 \cos \theta$

$$\text{Then } QS^2 = (ae^2 \cos \theta - ae)^2$$

$$= a^2 e^4 \cos^2 \theta - 2a^2 e^3 \cos \theta + a^2 e^2$$

If $QS = ePS$, then $QS^2 = e^2 PS^2 \Rightarrow \frac{QS^2}{e^2} = PS^2$

$$\frac{QS^2}{e^2} = \frac{a^2 e^4 \cos^2 \theta - 2a^2 e^3 \cos \theta + a^2 e^2}{e^2} = a^2 e^2 \cos^2 \theta - 2a^2 e \cos \theta + a^2$$

Set $\frac{QS^2}{e^2} = PS^2$:

$$a^2 e^2 \cos^2 \theta - 2a^2 e \cos \theta + a^2 = a^2 e^2 - 2a^2 e \cos \theta + a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

$$a^2 e^2 \cos^2 \theta + a^2 = a^2 e^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

Use $b^2 = a^2(1 - e^2) \Rightarrow a^2 e^2 = a^2 - b^2$:

$$(a^2 - b^2) \cos^2 \theta + a^2 = a^2 e^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

$$a^2 - b^2 \cos^2 \theta = a^2 e^2 + b^2 \sin^2 \theta$$

$$a^2 = a^2 e^2 + b^2$$

The last equation is true as it is a rearrangement of the defining equation for eccentricity, so this is proved.